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The reflection of a solitary wave by a vertical wall

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In this paper we consider the head-on collision of two equal solitary waves this being equivalent, in the absence of viscosity to the reflection of one solitary wave by a vertical wall. The perturbation expansion of the Euler equations, which lead to the Boussinesq equation at lowest order, is recast to obtain two weakly coupled KdV equations. We show analytically that the amplitude of the solitary wave after reflection is reduced. This change in amplitude is shown to be fifth order in ϵ, the amplitude of the wave. It is also shown that the experimentally observed transient loss of amplitude can be explained by the presence of the third-order dispersive tail.

1. Introduction

The earliest analytic approximations for the solitary wave were given by Boussinesq (1872) and Korteweg & de Vries (1895), who both derived equations for the propagation of water waves of small amplitude and long wavelength. The equation derived by Korteweg & de Vries, now called the KdV equation, was for unidirectional propagation of water waves, while the earlier work of Boussinesq derived an equation which allows waves that propagate in opposite directions. Byatt-Smith (1971) showed that the Boussinesq equation admitted solutions that consisted of two solitary waves travelling in opposite directions, with each solitary wave satisfying an appropriate KdV equation. He also showed that the interaction term was of smaller order than the waves themselves but was correctly predicted by the Boussinesq equation. Miles (1977) extended this result to the interaction of unsteady waves travelling in opposite directions and decomposed the interaction term into a transient term and a phase shift.

This pattern repeats itself at higher orders so that the nth-order travelling wave solution completely determines the $(n+1)$th-order interaction term. Su & Mirie (1980, 1982) completed this process to third order where it becomes apparent that the expansion produced is not uniformly valid for all times. The reason for this breakdown is that the method allows only for uniformly travelling waves and interaction terms, and does not allow for a distortion of the travelling waves apart from a possible phase shift. Su & Mirie (1980) demonstrated that at third order a dispersive tail must appear and found no change of amplitude. Their numerical computations (1982) confirmed their theoretical prediction of the dispersive tail and found an amplitude-dependent change in the amplitude of the waves. This change appeared to be of smaller order than the cube of the original amplitude of the wave but they were unable to determine its exact order of magnitude.

Fenton & Rienecker (1982) provide a Fourier-series method for solving the full Euler equations. Their method assumes periodic waves and is applicable to waves travelling in the same or opposite directions. They treat 'solitary' waves by looking at waves of long but finite wavelength. The present author believes that the restriction of periodicity limits the applications of that paper. A dispersive tail is essentially expressed as an integral over a continuous spectrum while the restriction of periodicity limits the form of solution to a sum over a discrete spectrum. This may account for the fact that their conclusions differ from those of Su & Mirie (1980, 1982). Fenton & Rienecker (1982) find that the wave amplitude alters at the third order although no dispersive tail is produced. However this result is incompatible with their claim that no energy is lost at least in the true solitary wave case. The reason for this apparent inconsistency is that they claim that after reflection the 'solitary waves' are smaller but faster. This may apply to periodically interacting waves but cannot be the case for solitary waves when they have reached their permanent form after interaction. This is because the wave speed is a monotonic increasing function of amplitude except for waves very close to the maximum. (See for example Byatt-Smith & Longuet-Higgins 1976 or Longuet-Higgins & Fenton 1974).

There are experimental observations that show that at least for small times after interaction the waves are smaller but faster (see Maxworthy 1976 and Renouard, Seabra-Santos & Temperville 1985). However, the later experiments show that this loss is only transient and that as the solitary wave progresses after reflection the 'loss' of amplitude and the original shape is recovered. Renouard *et al.* also include the effects of viscous damping on the amplitude of the solitary wave and conclude 'that the amplitude of the reflected solitary wave is the amplitude that the solitary wave would have if the wall did not exist and if viscous damping were acting alone'. Presumably this means that any loss of amplitude due to reflection was too small to be accurately measured. Thus the observed transient loss of amplitude must be due to the superposition of the reflected wave and the dispersive tail. For small times after reflection the dispersive tail must then appear as a wave of depression at least in the vicinity of the maximum elevation of the solitary wave.

In this paper we show that the loss of amplitude of a reflected solitary wave is of fifth order. We shall consider the case of two equal waves propagating in opposite directions. Starting from the Euler equations for fluid flow we derive interaction equations which are perturbations of the KdV equations. From these equations we analyse the formation of the dispersive tail and the subsequent loss of amplitude of the reflected solitary wave.

2. Basic equations

We consider unsteady, two-dimensional irrotational motion of a fluid. The motion is assumed to be such that all disturbances tend to zero at infinity where the depth, h, is uniform. It will be convenient to choose units so that

$$h = g = 1, \tag{2.1}$$

where g is the acceleration due to gravity.

Let (x,y) be horizontal and vertical coordinates, t the time, η the free-surface displacement and φ the velocity potential. The boundary-value problem is then described by

$$\varphi_{xx}+\varphi_{yy} = 0 \quad (0 < y < 1+\eta), \tag{2.2}$$

$$\varphi_y = 0 \quad (y = 0), \tag{2.3}$$

$$\eta_t+\varphi_x\eta_x-\varphi_y = 0 \quad (y = 1+\eta), \tag{2.4}$$

and
$$\eta+\varphi_t+\tfrac{1}{2}\varphi_y^2+\tfrac{1}{2}\varphi_y^2 = 0. \tag{2.5}$$

Following Miles (1977), Su & Mirie (1980) or Byatt-Smith (1987*a*, *b*) we look for a solution of the form
$$\varphi(x,y,t) = \sum_0^\infty (-\mathrm{D}^2)^n \frac{\Phi(x,t)\,y^{2n}}{(2n)!}, \tag{2.6}$$
where $\mathrm{D} = \partial/\partial x$.

In terms of η and $W(x,t) \equiv \partial\Phi/\partial x$, (2.4) and the x-derivative of (2.5) can be written as
$$\eta_t + W_x + \mathrm{D}\left\{\eta W + \sum_1^\infty (-1)^n \frac{(1+\eta)^{2n+1}}{(2n+1)!}\mathrm{D}^{2n}W\right\} = 0, \tag{2.7}$$
and
$$W_t+\eta_x+\mathrm{D}\left\{\tfrac{1}{2}W^2+\sum_{n=1}^\infty (-1)^n \frac{(1+\eta)^{2n}}{2n!}\left[\mathrm{D}^{2n-1}\partial_t W+\frac{1}{2}\sum_{m=0}^{2n}(-1)^m\binom{2n}{m}\mathrm{D}^m W \mathrm{D}^{2n-m}W\right]\right\} = 0, \tag{2.8}$$
where $\partial_t \equiv \partial/\partial t$,

By adding and subtracting these two equations they may be rewritten as
$$(\partial_t \pm \mathrm{D})(W\pm\eta)+\mathrm{D}F_\pm = 0, \tag{2.9}$$
where
$$F_\pm = \tfrac{1}{2}W^2 \pm W\eta + \sum_{n=1}^\infty (-1)^n \frac{(1+\eta)^{2n}}{2n!}\left[\partial_t \mathrm{D}^{2n-1}W \pm \frac{1+\eta}{2n+1}\mathrm{D}^{2n-1}W + \frac{1}{2}\sum_{m=0}^{2n}(-1)^m \binom{2n}{m}\mathrm{D}^m W \mathrm{D}^{2n-m}W\right]. \tag{2.10}$$

3. Perturbation equations for two equal solitary waves travelling in opposite directions

We consider two equal solitary waves initially far apart, of small but finite amplitude travelling towards each other. Provided the symmetry is maintained this corresponds, at least for inviscid flow, to the situation where a single solitary wave is travelling towards, and ultimately reflected by a vertical wall. This is the only case we discuss here. However, there is also the possibility that symmetry-breaking perturbations are unstable which, if present, would destroy the equivalence of the two problems. These have been shown to occur in periodic gravity waves by, for example, Saffman (1985), Zufiria & Saffman (1986) and Tanaka (1983, 1985) and also in solitary waves by Tanaka (1986) and Tanaka *et al.* (1987). However, in all of these cases the amplitude has to be large, often quite close to the maximum, before the symmetry-breaking perturbations become unstable.

In the absence of any interaction terms each solitary wave will be a function of a single phase variable and we introduce new coordinates
$$\xi_1 = \epsilon^{\frac{1}{2}}k(x-ct), \quad \xi_2 = \epsilon^{\frac{1}{2}}k(x+ct), \tag{3.1}$$
where $0<\epsilon \ll 1$ is a small dimensionless parameter representing the order of magnitude of the wave amplitude, k is the wavenumber and c the wave speed.

We introduce the notation
$$\partial_1 = \frac{\partial}{\partial \xi_1}, \quad \partial_2 = \frac{\partial}{\partial \xi_2}, \tag{3.2}$$

and the change of dependent variables

$$\alpha = \tfrac{1}{2}\epsilon^{-1}(W+\eta), \quad \beta = \tfrac{1}{2}\epsilon^{-1}(\eta - W). \tag{3.3}$$

In terms of these variables (2.9) becomes

$$4c\epsilon\, \partial_2 \alpha + (\partial_1 + \partial_2)\tilde{F}_+ = 0, \tag{3.4}$$

and

$$4c\epsilon\, \partial_1 \beta + (\partial_1 + \partial_2)\tilde{F}_- = 0, \tag{3.5}$$

where now

$$\begin{aligned}\tilde{F}_\pm &= \mp 2(c-1)\,\epsilon\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix} + \tfrac{1}{2}\epsilon^2(\alpha-\beta)^2 \pm \epsilon^2(\alpha^2-\beta^2)\\ &+ \sum_{n=1}^{\infty}(-1)^n \frac{(1+\epsilon(\alpha+\beta))^{2n}}{2n!}\Big[c\epsilon^{n+1}k^{2n}(\partial_2-\partial_1)(\partial_1+\partial_2)^{2n-1}(\alpha-\beta)\\ &\pm \frac{(1+\epsilon(\alpha+\beta))}{2n+1}\epsilon^{n+1}k^{2n}(\partial_1+\partial_2)^{2n}(\alpha-\beta)\\ &+ \tfrac{1}{2}\epsilon^{n+2}k^{2n}\sum_{m=0}^{2n}(-1)^m\binom{2n}{m}(\partial_1+\partial_2)^m(\alpha-\beta)(\partial_1+\partial_2)^{2n-m}(\alpha-\beta)\Big].\end{aligned} \tag{3.6}$$

The procedure adopted by Su & Mirie is to expand the parameter c and the variables α and β as a power series in ϵ. They also allow for a phase shift by making an additional expansion of the independent variables. However, their expansion is still not uniformly valid and breaks down as $t \to +\infty$, that is after the interaction has taken place. The result of this breakdown is that the travelling wave part of the solution, after interaction, no longer satisfies the equation for a wave propagating without change of speed and shape. The reason for the breakdown is that during and after interaction the travelling wave part of the solution is distorted on a timescale that is large compared with the timescale, t, based on the wave speed. Su & Mirie tacitly use this argument and set up an initial-value problem for the distorted travelling wave after interaction. However, there is no real origin of time for this initial-value problem and their solution for the dispersive tail produced by the interaction is valid only for large time.

We propose to model, more correctly, this interaction by deriving an equation for the unsteady travelling wave part of the solution. This will take the form of an approximate KdV equation which will be analysed by the method of inverse scattering. We do not propose to rederive the full third-order solution which has been given for example by Chappelear (1962), Grimshaw (1971), Fenton (1972) and Su & Mirie (1980). Instead we propose to pick out the term that gives rise to the distortion of the travelling wave and treat this term by itself. Thus for example the terms of order ϵ in (3.4) and (3.5) give

$$4c\,\partial_2\alpha = 4c\,\partial_1\beta = 0, \tag{3.7}$$

provided

$$c = 1 + O(\epsilon). \tag{3.8}$$

If we use $\partial_2\alpha = O(\epsilon)$ the terms of order ϵ^2 in (3.4) give

$$4\,\partial_2\alpha + \epsilon(\partial_1^3\alpha + 3\alpha\,\partial_1\alpha - \partial_1\alpha) = \epsilon[(\partial_1\alpha)\beta + \partial_2(\alpha\beta) + \partial_2(-\tfrac{1}{2}\beta^2 + 2\,\partial_2^2\beta)], \tag{3.9}$$

provided

$$c_1 = \tfrac{1}{2},\ k^2 = 3. \tag{3.10}$$

The latter conditions (3.10) are required to eliminate the secular terms that arise in the expansion (see Su & Mirie (1980) for further details).

If we now look for a solution

$$\alpha = \alpha_0(\xi_1 + \epsilon\theta_1(\xi_2)) + \epsilon\alpha_1(\xi_1, \xi_2), \tag{3.11}$$

Then $\alpha_0(\chi_1)$ satisfies the ordinary differential equation

$$\alpha_0''' + 3\alpha_0\alpha_0' = 0, \tag{3.12}$$

which has the solution

$$\alpha_0 = \operatorname{sech}^2(\tfrac{1}{2}\chi_1). \tag{3.13}$$

The solution for β_0 is obtained similarly and then (3.9) can be satisfied by choosing θ_1 and α_1 as

$$\theta_1 = \int^{\xi_2} \beta_0(\xi_0)\,\mathrm{d}\xi_0, \tag{3.14}$$

$$\begin{aligned}\alpha_1 &= \tfrac{1}{4}\alpha_0\beta_0 - \tfrac{1}{8}\beta_0^2 + \tfrac{1}{2}\beta_0'' + F_1(\xi_1)\\ &= \tfrac{1}{4}\alpha_0\beta_0 - \tfrac{7}{8}\beta_0^2 + \tfrac{1}{2}\beta_0 + F_1(\xi_1),\end{aligned} \tag{3.15}$$

so that

$$\chi_1 = \xi_1 + \epsilon\int^{\xi_2} \beta_0(\xi_0)\,\mathrm{d}\xi_0. \tag{3.16}$$

The alternative is to use (3.11) and (3.16) to show, without solving (3.9), that there exists a transformation from (α, ξ_1) to $(\tilde{\alpha}, \chi_1)$ so that (3.9) is transformed into

$$4\frac{\partial\tilde{\alpha}}{\partial\xi_2} + \epsilon\left(\frac{\partial^3\tilde{\alpha}}{\partial\chi_1^3} + 3\alpha\frac{\partial\tilde{\alpha}}{\partial\chi_1} - \frac{\partial\tilde{\alpha}}{\partial\chi_1}\right) = 0 \quad \text{to} \quad O(\epsilon^2). \tag{3.17}$$

We do not wish to find this transformation but use its existence to justify studying the solution of (3.9) when the right-hand side is put equal to zero.

At the next order of ϵ, retaining only the terms that lead to singular behaviour, we obtain

$$4\,\partial_2\alpha + \epsilon(\partial_1^3\alpha + 3\alpha\,\partial_1\alpha - \partial_1\alpha) = 9\epsilon^2\alpha\,\partial_1\alpha\beta. \tag{3.18}$$

Similarly from (3.5) we obtain

$$4\,\partial_1\beta + \epsilon(\partial_2^3\beta + 3\beta\,\partial_2\beta - \partial_2\beta) = 9\epsilon^2\beta\,\partial_2\beta\alpha. \tag{3.19}$$

We transform (3.18) and (3.19) to approximate KdV equations by writing

$$\alpha = -2u, \beta = -2v, \quad \tau_i = \tfrac{1}{2}\epsilon\xi_i, \quad i = 1, 2. \tag{3.20}$$

In terms of these variables (3.18) and (3.19) are

$$\frac{\partial u}{\partial\tau_2} = 3u\,\partial_1 u - \tfrac{1}{2}\partial_1^3 u + \tfrac{1}{2}\partial_1 u + 18\epsilon u\,\partial_1 uv \equiv \mathrm{X}_1(u) + \tfrac{1}{2}\mathrm{D}_1(u) + \epsilon\mathrm{X}_1^1(u, v), \tag{3.21}$$

and

$$\frac{\partial v}{\partial\tau_1} = 3v\,\partial_2 v - \tfrac{1}{2}\partial_2^3 v + \tfrac{1}{2}\partial_2 v + 18\epsilon v\,\partial_2 vu \equiv \mathrm{X}_2(v) + \tfrac{1}{2}\mathrm{D}_2(v) + \epsilon\mathrm{X}_2^1(v, u), \tag{3.22}$$

here

$$\mathrm{D}_1 \equiv \partial/\partial\xi_1, \quad \mathrm{X}_1 = -\tfrac{1}{2}\mathrm{D}_1^3 + u\,\mathrm{D}_1 + \mathrm{D}_1\,u, \tag{3.23}$$

represent the first two operators in the hierarchy of KdV flows (see McKean & Van Morbeke 1975 for example). X_2 is defined similarly in terms of v with $\mathrm{D}_2 \equiv \partial/\partial\xi_2$. This notation has been chosen to coincide with that of Byatt-Smith (1987*a*, *b*) who studied the interaction of two solitary waves moving in the same direction.

4. The interaction of two solitary waves

Equations (3.21) and (3.22) are both perturbations of the KdV equation which are connected via the perturbation terms X_1^1 and X_2^1. These equations may be solved using the method of perturbed inverse scattering developed by Karpman & Maslov (1977*a–c*), Keener and McLaughlin (1977) and Kaup and Newell (1978) and used by Byatt-Smith (1987*a,b*). In certain circumstances they may be solved by the method of Su & Mirie (1980) who follow the suggestion of Jeffrey & Kakutani (1970).

We start by deriving the general solitary wave solution of the unperturbed equation

$$u_{\tau_2} = X_1(u)+\tfrac{1}{2}D_1(u). \tag{4.1}$$

This by hypothesis is a function of the single variable

$$\theta = \kappa\xi_1-\tfrac{1}{2}(\kappa^3-\kappa)\,\tau_2, \tag{4.2}$$

and takes the form

$$\tilde{u}_0(\theta) = -\tfrac{1}{2}\kappa^2 \operatorname{sech}^2\tfrac{1}{2}\theta. \tag{4.3}$$

In our case we have defined coordinates in which the unperturbed wave before interaction is stationary so that $\kappa = 1$. We denote this particular solution by u_0 so that

$$u_0(\xi_1) = -\tfrac{1}{2}\operatorname{sech}^2\tfrac{1}{2}\xi_1. \tag{4.4}$$

The method of Su & Mirie is to write u as

$$u = u_0(\xi_1)+u_1(\xi_1,\tau_2), \tag{4.5}$$

with a similar expansion for v, and solve the linearized equation

$$\frac{\partial u_1}{\partial\tau_2} = 3\,\partial_1(u_1u_0)-\tfrac{1}{2}\partial_1^3u_1+\tfrac{1}{2}\partial_1u_1+18\epsilon u_0\,\partial_1u_0\,v_0, \tag{4.6}$$

where $v_0 = -\tfrac{1}{2}\operatorname{sech}^2\tfrac{1}{2}\xi_2$.

The solution of (4.6) was first outlined by Jeffrey & Kakutani (1970) and is obtained by variation of parameters from the solution $U_\mu(\xi_1,\tau_2)$ of the homogeneous equation. This is given by

$$U_\mu(\xi_1,\tau_2) = F_1(\xi_1,k)\,e^{-\frac{1}{2}i\mu\tau_2+ik\xi_1}, \tag{4.7}$$

where

$$\mu = -(k+k^3), \tag{4.8}$$

and

$$F_1(\xi_1,k) = ik(k^2-1)-4iku_0-2\,\partial_1u_0+2k^2\,\partial_1u_0/u_0. \tag{4.9}$$

The solution of (4.6) can then be expressed as

$$u_1(\xi_1,\tau_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\tau_2} A(k,\tau_0)\,F_1(\xi_1,k)\,e^{-\frac{1}{2}i\mu(\tau_2-\tau_0)+ik\xi_1}\,d\tau_0\,dk, \tag{4.10}$$

where

$$\int_{-\infty}^{\infty} A(k,\tau_2)\,F_1(\xi_1,k)\,e^{ik\xi_1}\,dk = 18\epsilon u_0\,\partial_1u_0\,v_0. \tag{4.11}$$

The bottom limit of the τ_0 integration is determined from the condition that before interaction $(t\rightarrow-\infty)\,u_1$ is zero.

Then we may write

$$A(k,\tau_2) = B(k)\,\epsilon v_0(2\tau_2/\epsilon), \tag{4.12}$$

where

$$-\left(\frac{d^3}{d\xi_1^3}-\frac{\partial_1u_0}{u_0}\frac{d^2}{d\xi_1^2}-(4u_0+1)\frac{d}{d\xi_1}-2\,\partial_1u_0\right)\int_{-\infty}^{\infty}B(k)\,e^{ik\xi_1}\,dk = 18u_0\,\partial_1u_0. \tag{4.13}$$

Su & Mirie (1980) make the statement 'Since $F_1(\xi_1, k)$ is bounded for all ξ_1, to have a bounded solution $U_\mu(\xi_1, \tau_2)$ in ξ_1 we must take k real which in turn requires a real μ. Therefore the general solution u_1 (4.10) can be expressed as an integral over all real k.' This is not true under all circumstances although it does apply in their solution where the perturbation produces only a dispersive tail and a phase shift. Such perturbations must satisfy the requirement that

$$\int_{-\infty}^{\infty} u_0(\xi_1)\,\mathrm{X}_1^1(u_0(\xi_1), v_0(\xi_2))\,\mathrm{d}\xi_1 \equiv 0. \tag{4.14}$$

For arbitrary perturbations X_1^1 expressible in the form $\mathrm{X}_1^1 = U_0(\xi_1)\,V_0(\xi_2)$ and also satisfying (4.14), the solution of (4.13) can be written as

$$\int_{-\infty}^{\infty} B(k)\,\mathrm{e}^{\mathrm{i}k\xi_1}\,\mathrm{d}k = -\frac{1}{u_0(\xi_1)}\int_{-\infty}^{\xi_1} u_0(\zeta_1)\,\mathrm{d}\zeta_1 \int_{\zeta_4}^{\zeta_1} \frac{\mathrm{d}\zeta_2}{u_0(\zeta_2)} \int_{-\infty}^{\zeta_2} u_0(\zeta_3)\,U_0(\xi_3)\,\mathrm{d}\zeta_3, \tag{4.15}$$

where ζ_4 is a constant chosen so that the expression converges as $\xi_1 \to +\infty$.

However if (4.14) is not satisfied then the rate of change of amplitude is given by the equation

$$\frac{\mathrm{d}\kappa}{\mathrm{d}\tau_2} = \frac{1}{8}\int_{-\infty}^{\infty} u_0(\xi_1)\,\mathrm{X}_1^1(u_0, v_0)\,\mathrm{d}\xi_1, \tag{4.16}$$

where κ is defined via (4.3). This expression has been derived by Byatt-Smith (1987*a*, *b*).

For the case $\mathrm{X}_1^1 = \partial_1 u_0 v_0$ it is easily verified that (4.14) holds and with $\zeta_4 = 0$ we obtain

$$\int_{-\infty}^{\infty} B(k)\,\mathrm{e}^{\mathrm{i}k\xi_1}\,\mathrm{d}k = 2 - u_0(\xi_1), \tag{4.17}$$

so that

$$B(k) = 2\delta(k) - \bar{u}_0(k), \tag{4.18}$$

where

$$\bar{u}_0(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}k\xi_1}\,u_0(\xi_1)\,\mathrm{d}\xi_1 = -\frac{k}{\sinh \pi k}. \tag{4.19}$$

Hence

$$\begin{aligned} u_1(\xi_1, \tau_2) &= \epsilon \int_{-\infty}^{\infty}\int_{-\infty}^{\tau_2} v(2\tau_0/\epsilon)\,(2\delta(k) - \bar{u}_0(k))\,F_1(\xi_1, k)\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}\mu(\tau_2-\tau_0)+\mathrm{i}k\xi_1}\,\mathrm{d}\tau_0\,\mathrm{d}k \\ &= -2\epsilon^2\,\partial_1 u_0 \int_{-\infty}^{\xi_2} v_0(\xi_0)\,\mathrm{d}\xi_0 \\ &\quad -\tfrac{1}{2}\epsilon^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\xi_2} v_0(\xi_0)\,\bar{u}_0(k)\,F_1(\xi_1, k)\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\mu\epsilon(\xi_2-\xi_0)+\mathrm{i}k\xi_1}\,\mathrm{d}\xi_0\,\mathrm{d}k. \end{aligned} \tag{4.20}$$

If $\xi_2 \to \infty$ with τ_2 fixed then (see (6.1)–(6.3))

$$u_1(\xi_1, \tau_2) = 4\epsilon^2\,\partial_1 u_0 + \epsilon^2 \int_{-\infty}^{\infty} \bar{u}_0(k)\,F_1(\xi_1, k)\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}\mu\tau_2+\mathrm{i}k\xi_1}\,\mathrm{d}k. \tag{4.21}$$

This result is identical to that obtained by Su & Mirie, which is thus only valid in this limit.

The first term in (4.20) can be interpreted as a phase shift using the same arguments as in §3. Alternatively this phase shift can be eliminated by adding to the right-hand side of (4.6) a term equal to $4\,\partial_1 u_0 v_0$. The phase shift must be properly interpreted as such to proceed with the next approximation and, following the philosophy of §3, we take the latter course of action. This results in a revised first approximation u_1 given by

$$u_1(\xi_1, \tau_2) = -\tfrac{1}{2}\epsilon^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\xi_2} v_0(\xi_0)\, \bar{u}_0(k)\, F_1(\xi_1, k)\, \mathrm{e}^{-\frac{1}{4}\mathrm{i}\mu\epsilon(\xi_2-\xi_0)+\mathrm{i}k\xi_1}\, \mathrm{d}\xi_0\, \mathrm{d}k. \tag{4.22}$$

This term now represents the dispersive tail and is clearly of order ϵ^2 and for convenience we write $\tilde{u}_1 = \epsilon^{-2}u_1$. However, it should be noted that $\tilde{u}_1 \equiv \tilde{u}_1(\xi_1, \tau_2, \epsilon)$, which cannot be uniformly expanded as a power series in ϵ without the formal introduction of the second timescale ξ_2. However, we do note that from (4.21) that in the limit $\xi_2 \to \infty$ with τ_2 fixed $\tilde{u}_1$ is independent of ϵ.

The advantage of this method of solution is that (4.22) is obtained readily without the scattering–inverse scattering recipe. The difficulties that can arise are easily understood by comparing the method with the perturbed inverse scattering method. This can easily be achieved by first defining the function f as

$$f(\xi_1, k) = \{\mathrm{i}k - \partial_1 u_0(\xi_1)/u_0(\xi_1)\}\, \mathrm{e}^{\frac{1}{2}\mathrm{i}k\xi_1}. \tag{4.23}$$

This function can easily be shown to satisfy

$$\frac{\mathrm{d}}{\mathrm{d}\xi_1} f^2 = -U_\mu(\xi_1, 0) = -F_1(\xi_1, k)\, \mathrm{e}^{\mathrm{i}k\xi_1}. \tag{4.24}$$

With k real f represents the eigenfunction of the continuous spectrum of the operator

$$\mathrm{Q} = -\mathrm{D}_1^2 + u_0, \tag{4.25}$$

with f satisfying

$$\mathrm{Q}(f) = \lambda f. \tag{4.26}$$

For the given value $u_0(\xi_1)$ there is also one additional discrete eigenvalue $\lambda = -1$ and $f(\xi_1, \mathrm{i})$ also represents the eigenfunction for this eigenvalue.

The discrete eigenvalue and the quantity $\tilde{c}^2$ defined by

$$\tilde{c}^2 \int_{-\infty}^{\infty} |f^2(\xi_1, \mathrm{i})|\, \mathrm{d}\xi_1 = 1$$

form part of the spectral data and their variation in the perturbed system allows for a uniformly valid approximation with a change of amplitude and phase shift. Equation (4.16) can then be identified with that obtained by Byatt-Smith (1987*a*) by noting the relation $f^2(\xi_1, \mathrm{i}) = -2u_0(\xi_1)$. Knowledge of the change of the remaining parts of the spectral data is required for the computation of the altered solitary wave and the dispersive tail that is produced by the perturbation. (Again, see Karpman & Maslov 1977*a–c*; Keener & McLaughlin 1977; Kaup & Newell 1978; Byatt-Smith 1987*a, b*.) This backward or inverse scattering is achieved via the Gelfand–Levitan integral equation (see Gelfand & Levitan 1951 or Agranovitch & Marchenko 1963). Our main aim is to determine the change of amplitude of the solitary waves during interaction. In order to do this we need to proceed to the next approximation. However, it only requires the appropriate form of (4.16) and it is unnecessary to determine the correction to the dispersive tail.

5. Calculation of the change of amplitude

To discuss the change of amplitude we first need to look at the higher-order terms in the expansion of (3.5) and (3.6) which give rise to further terms on the right-hand side of (3.21) and (3.22). Again we assume that these terms fall into two categories.

In the first category there are those that arise naturally when a single wave is expanded to high orders and also higher-order interaction terms corresponding to the expansion of α_1 (see (3.11)). These represent the uniformly valid terms that arise in the expansion of the wave interaction problem and we assume that there exists a similar transformation to new variables which remove these terms (see (3.17)). The remaining terms are the ones that produce distortion of the wave profile but do not lead, to lowest order, to a change of amplitude. The effect of these terms is to modify the right-hand side of (4.6). When this is solved by the method of §4, the linearized solution will have a higher-order correction to the dispersive tail but no change in amplitude. Thus the change of amplitude first appears when (4.6) is solved correct to second order as outlined below. The leading term in this second-order solution will clearly come from the product terms involving $\tilde{u}_1$ and $\tilde{v}_1$ arising from the order-ϵ terms in (3.21) and (3.22) and not from the higher-order terms. Thus we now wish to solve the equations

$$\frac{\partial u}{\partial \tau_2} = 3u\,\partial_1 u - \tfrac{1}{2}\partial_1^3 u + \tfrac{1}{2}\partial_1 u + \epsilon(18u+4)\,\partial_1 uv, \tag{5.1}$$

and

$$\frac{\partial v}{\partial \tau_1} = 3\,\partial_2 v - \tfrac{1}{2}\partial_2^3 v + \tfrac{1}{2}\partial_2 v + \epsilon(18v+4)\,\partial_2 vu, \tag{5.2}$$

correct to second order.

The first approximation to (5.1) derived in §4 is

$$u = u_0(\xi_1) + \epsilon^2\tilde{u}_1(\xi_1, \tau_2, \epsilon), \tag{5.3}$$

where

$$u_0 = -\tfrac{1}{2}\operatorname{sech}^2 \tfrac{1}{2}\xi_1, \tag{5.4}$$

and

$$\tilde{u}_1 = -\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\xi_2} v_0(\xi_0)\,\bar{u}_0(k)\,F_1(\xi_1, k)\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\mu\epsilon(\xi_2-\xi_0)+\mathrm{i}k\xi_1}\,\mathrm{d}\xi_0\,\mathrm{d}k. \tag{5.5}$$

The first approximation to (5.2) is obtained in a similar fashion and

$$\tilde{v}_1 = +\frac{1}{2}\int_{-\infty}^{\infty}\int_{\xi_1}^{\infty} u_0(\xi_0)\,\bar{v}_0(k)\,F_2(\xi_2, k)\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\mu\epsilon(\xi_1-\xi_0)+\mathrm{i}k\xi_2}\,\mathrm{d}\xi_0\,\mathrm{d}k. \tag{5.6}$$

Again the upper limit of the ξ_0 integration is determined by the condition that v_1 is zero before interaction.

We now proceed to the next approximation by writing

$$u = u_0(\xi_1) + \epsilon^2\tilde{u}_1(\xi_1, \tau_2, \epsilon) + \epsilon^4\tilde{u}_2(\xi_1, \tau_2, \epsilon), \tag{5.7}$$

with a similar expansion for v.

Introducing this expansion into (5.1) and linearizing with respect to $\tilde{u}_2$ yields the equation

$$\frac{\partial \tilde{u}_2}{\partial \tau_2} = 3\,\partial_1(\tilde{u}_2 u_0) - \tfrac{1}{2}\partial_1^3\tilde{u}_2 + \tfrac{1}{2}\partial_1\tilde{u}_2 + \epsilon^{-1}\{(18u_0+4)\,\partial_1 u_0\tilde{v}_1 + \{\partial_1(18u_0+4)\ \tilde{u}_1\}\,\tilde{v}_0\} + 3\,\partial_1(\tilde{u}_1^2). \tag{5.8}$$

The last term gives an order-ϵ term to $\tilde{u}_2$ so that from the analogue of (4.16) the change of amplitude is given, to leading order, by

$$\begin{aligned}\frac{\mathrm{d}\kappa}{\mathrm{d}\tau_2} &= \tfrac{1}{8}\epsilon^4\int_{-\infty}^{\infty} u_0\,\epsilon^{-1}\{(18u_0+4)\,\partial_1 u_0\tilde{v}_1 + \partial_1(18u_0+4)\,\tilde{u}_1)\,v_0\}\,\mathrm{d}\xi_1 \\ &= \tfrac{1}{8}\epsilon^3\int_{-\infty}^{\infty}\{u_0(18u_0+4)\,\partial_1 u_0\tilde{v}_1 - (18u_0+4)\,\partial_1 u_0\tilde{u}_1 v_0\}\,\mathrm{d}\xi_1.\end{aligned} \tag{5.9}$$

Following the arguments of Byatt-Smith (1987*a*, *b*) we calculate the total change in κ as (see the Appendix)

$$[\kappa] = \int_{-\infty}^{\infty} \frac{\mathrm{d}\kappa}{\mathrm{d}\tau_2}\mathrm{d}\tau_2 = \tfrac{1}{2}\epsilon \int_{-\infty}^{\infty} \frac{\mathrm{d}\kappa}{\mathrm{d}\tau_2}\mathrm{d}\xi_2 = -\frac{5\epsilon^4}{42} + O(\epsilon^5). \tag{5.10}$$

Using the definition of the individual waves (3.3) and the change of variable (3.20) the unscaled amplitude, a, of each wave is given by

$$a = \epsilon\kappa^2,$$

so that

$$[a] = 2\epsilon[\kappa] = -\frac{5\epsilon^5}{21} + O(\epsilon^6). \tag{5.11}$$

This calculation is consistent with the idea that the change in amplitude is due to the production of a dispersive tail. To leading order the energy of a solitary wave is

$$\begin{aligned} E_{\mathrm{s}} &= \int_{-\infty}^{\infty} \eta^2\,\mathrm{d}x = \int_{-\infty}^{\infty} 4\epsilon^2 u_0^2\, \epsilon^{-\frac{1}{2}}\mathrm{d}\xi = \mathrm{const} \times \epsilon^{\frac{3}{2}} \\ &= E_0\, a^{\frac{3}{2}}, \end{aligned} \tag{5.12}$$

since a is of order ϵ. Thus a change of amplitude results in a corresponding change in E_{s} given by

$$[E_{\mathrm{s}}] = \frac{3E_0}{2} a^{\frac{1}{2}}[a]. \tag{5.13}$$

On the assumption that the solitary wave and the dispersive tail are well separated the energy in the dispersive tail is

$$E_{\mathrm{t}} = \int_{-\infty}^{\infty} 4(\epsilon^3 \tilde{u}_1)^2\, \epsilon^{-\frac{1}{2}}\mathrm{d}\xi = \mathrm{const} \times \epsilon^{\frac{11}{2}} = E_1\, a^{\frac{11}{2}}. \tag{5.14}$$

Thus

$$[a] = -\frac{2E_1}{3E_0} a^5 = O(\epsilon^5). \tag{5.15}$$

This simple energy balance can also be applied in the case of two solitary waves travelling in the same direction. This shows that the net energy in the two solitary waves after interaction is unaltered to leading order despite the alteration in amplitude of the two waves. This result was obtained by Byatt-Smith (1987*b*), who calculated these changes of amplitude. However, his remarks at the end of §4 of his paper, about the order of magnitude of each dispersive tail, as a consequence of this energy conservation, prove to be incorrect.

If the unscaled amplitudes of the two waves before interaction are a_1 and a_2 then the corresponding energy of each wave is given by (5.12), namely

$$E_{1\mathrm{s}} = E_0\, a_1^{\frac{3}{2}}, \quad E_{2\mathrm{s}} = E_0\, a_2^{\frac{3}{2}}. \tag{5.16}$$

Thus the change in the total energy of the two waves only is

$$[E_{1\mathrm{s}} + E_{2\mathrm{s}}] = \tfrac{3}{2}E_0(a_1^{\frac{1}{2}}[a_1] + a_2^{\frac{1}{2}}[a_2]). \tag{5.17}$$

If ϵ is the order of magnitude of the amplitude of the waves then Byatt-Smith (1987*b*) proves that the change of amplitude of each wave is of order ϵ^2 while the quantity $a_1^{\frac{3}{2}} + a_2^{\frac{3}{2}}$ is conserved to leading order so that

$$[E_{1\mathrm{s}} + E_{2\mathrm{s}}] = o(\epsilon^{\frac{5}{2}}). \tag{5.18}$$

This can be seen to be consistent with the production of dispersive tails whose order of magnitude is ϵ^2 and not ϵ^3 as Byatt-Smith (1987*b*) suggests. If we assume that the order of magnitude of the dispersive tails is ϵ^2 then following the argument used to obtain (5.14) we obtain

$$E_t = \text{const} \times \epsilon^{\frac{7}{2}}, \tag{5.19}$$

where E_t is the combined energy in the dispersive tails. Energy conservation then shows that

$$[E_{1s} + E_{2s}] = O(\epsilon^{\frac{7}{2}}), \tag{5.20}$$

which is in agreement with (5.18).

6. The transient change in amplitude during and after reflection

The experimental measurements of Maxworthy (1976) and Renouard *et al.* (1985) both show that immediately after reflection there is a transient loss of amplitude in the reflected wave. The latter experiments also show that this loss is recovered after a sufficient time has elapsed. We now show that this observation can be explained by the presence of the dispersive tail. This dispersive tail is usually thought of as a train of waves trailing behind the solitary wave. The amplitude of the train of waves decreases exponentially with distance, that is as $\xi_1 \to -\infty$, but because of dispersion decreases algebraically with time, that is as $\tau_2 \to \infty$. However, in addition to the train of waves there is also a wave of depression which occurs between the leading wave of the dispersive tail and the solitary wave. The amplitude of this wave decreases exponentially with time but only on the time-scale τ_2. Thus there is a slow decrease of amplitude of the wave of depression.

From (5.5) the scaled third-order dispersive tail, $\tilde{u}_1$, is given by

$$\tilde{u}_1 = -\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\xi_2} v_0(\xi_0)\,\bar{u}_0(k)\,F_1(\xi_1,k)\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\mu\epsilon(\xi_2-\xi_0)+\mathrm{i}k\xi_1}\,\mathrm{d}\xi_0\,\mathrm{d}k. \tag{6.1}$$

After the reflection the leading contribution is obtained by letting $\xi_2 \to \infty$ with τ_2 fixed so that

$$\tilde{u}_1 = -\frac{1}{2}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} v_0(\xi_0)\,\mathrm{e}^{-\frac{1}{4}\mathrm{i}\mu\epsilon\xi_0}\,\mathrm{d}\xi_0\right)\bar{u}_0(k)\,F_1(\xi_1,k)\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}\mu\tau_2+\mathrm{i}k\xi_1}\,\mathrm{d}k. \tag{6.2}$$

Since $v_0(\xi_0)$ is exponentially small as $\xi_0 \to \infty$, and $\bar{u}_0(k)$ is exponentially small as $k \to \infty$, this expression is unchanged to order ϵ if we replace $\mathrm{e}^{-\frac{1}{4}\mathrm{i}\mu\epsilon\xi_0}$ by 1. Thus

$$\tilde{u}_1(\xi_1,\tau_2) = \int_{-\infty}^{\infty} \bar{u}_0(k)\,F_1(\xi_1,k)\,\mathrm{e}^{\frac{1}{2}\mathrm{i}(k+k^3)\tau_2+\mathrm{i}k\xi_1}\,\mathrm{d}k + O(\epsilon). \tag{6.3}$$

Su & Mirie expand this integral as $\tau_2 \to \infty$ using the method of steepest descents to obtain

$$\tilde{u}_1(\xi_1,\tau_2) \sim \left(\frac{8\pi}{3|\kappa_0|\,\tau_2}\right)^{\frac{1}{2}}\bar{u}_0(k_0)\left\{\left(\frac{2k_0^2}{u_0(\xi_1)}-2\right)\partial_1 u_0(\xi_1)\cos(\tau_2 k_0^3 - \tfrac{1}{4}\pi) + k_0(k_0^2 - 1 - 4u_0(\xi_1))\sin(\tau_2 k_0^3 - \tfrac{1}{4}\pi)\right\}, \tag{6.4}$$

where $3k_0^2 = -(2\xi_1/\tau_2 + 1)$.

This is of course only valid if k_0 is real and not small; this requires ξ_1 to be less than $-\frac{1}{2}\tau_2$. As k_0^2 passes through zero the two saddle points, at $\pm k_0$, of the steepest descent integral coalesce and move up the imaginary axis. This changes the asymptotic

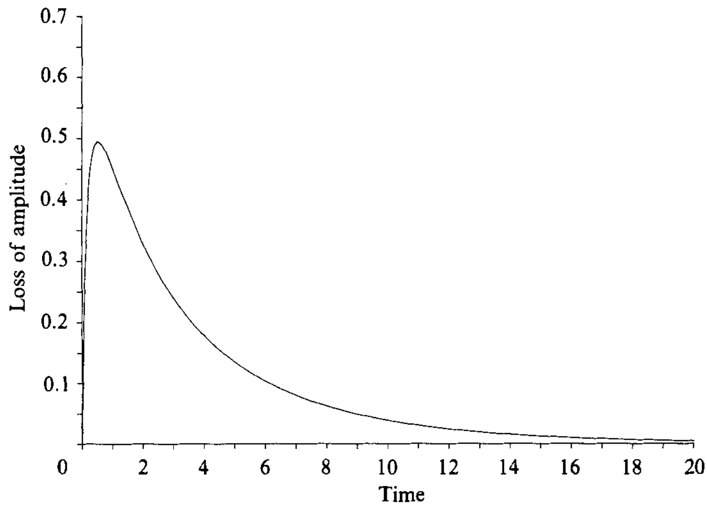


FIGURE 1. The transient loss of amplitude defined by (6.11) with the factor ϵ^3 being omitted.

behaviour of the integral from oscillatory to exponentially damped behaviour. Thus for $\xi_1 > -\frac{1}{2}\tau_2$

$$\tilde{u}_1(\xi_1, \tau_2) \sim \left(\frac{2\pi}{3k_1\tau_2}\right)^{\frac{1}{2}} \bar{u}_0(\mathrm{i}k_1)\left\{k_1(k_1^2+1)+4k_1 u_0 - 2\,\partial_1 u_0\left(1+\frac{k_1^2}{u_0}\right)\right\}\mathrm{e}^{-\tau_2 k_1^3}, \tag{6.5}$$

where
$$k_1 = |k_0| = (2\xi_1/\tau_2+1)^{\frac{1}{2}}/\sqrt{3}. \tag{6.6}$$

From (4.19) we have
$$\bar{u}_0(\mathrm{i}k_1) = -k_1 \operatorname{cosec} \pi k_1 < 0 \tag{6.7}$$

since $k_1 < 1$, at least for $\xi_1 < \tau_2$. Hence $\tilde{u}_1$ is positive, corresponding to a wave of negative depression provided

$$k_1^3 + k_1 - 2k_1 \operatorname{sech}^2 \tfrac{1}{2}\xi_1 - \tanh \tfrac{1}{2}\xi_1 \,(\operatorname{sech}^2 \tfrac{1}{2}\xi_1 - 2k_1^2) < 0. \tag{6.8}$$

In the limit as $\tau_2 \to \infty$ this becomes

$$(3 \tanh^2 \tfrac{1}{2}\xi_1 - 1)(2 + \sqrt{3} \tanh \tfrac{1}{2}\xi_1) < 0. \tag{6.9}$$

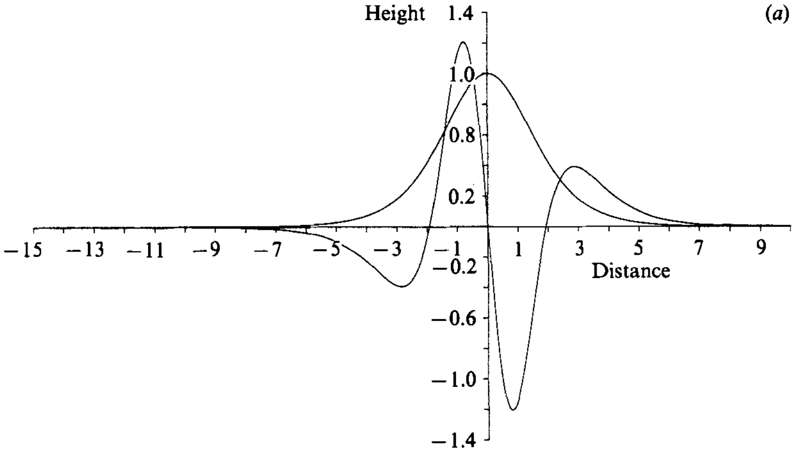
This inequality is satisfied only for ξ_1 in the range

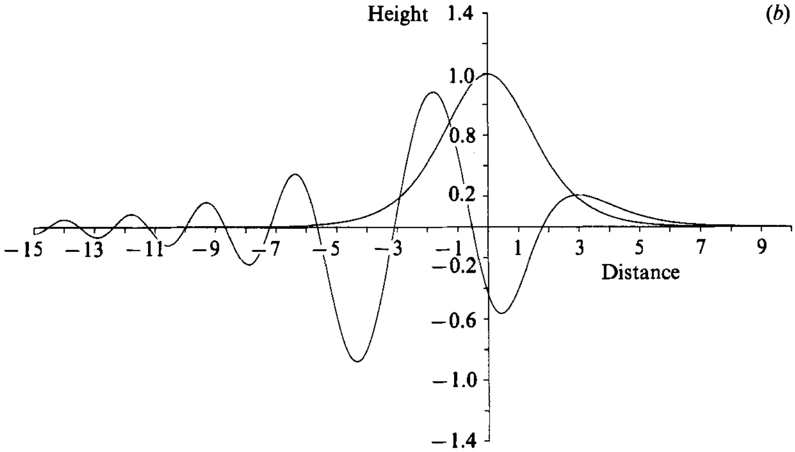
$$-2 \tanh^{-1}(1/\sqrt{3}) = -1.317 < \xi_1 < 1.317 = 2 \tanh^{-1}(1/\sqrt{3}). \tag{6.10}$$

Both asymptotic expansions (6.4) and (6.5) are invalid as $\xi_1 \to -\frac{1}{2}\tau_2$. To obtain a uniformly valid expansion we would need to express both expansions in terms of Airy functions using the method of two nearly coincident saddle points (see for example Bleistein & Handelsman 1975, chapter 9.2).

The transient loss of amplitude, $\delta a(\tau_2)$ is given by the expression

$$\begin{aligned} \delta a(\tau_2) &= \epsilon - \epsilon\alpha(0, \tau_2) \\ &= 2\epsilon^3 \tilde{u}_1(0, \tau_2) \\ &= 2\epsilon^3 \int_{-\infty}^{\infty} \bar{u}_0(k)\, \mathrm{i}(k+k^3)\, \mathrm{e}^{\frac{1}{2}\mathrm{i}(k+k^3)\tau_2}\, \mathrm{d}k \\ &= 4\epsilon^3 \int_0^{\infty} k^2(1+k^2) \operatorname{cosech}(\pi k) \sin \tfrac{1}{2}(k+k^3)\,\tau_2\, \mathrm{d}k. \end{aligned} \tag{6.11}$$





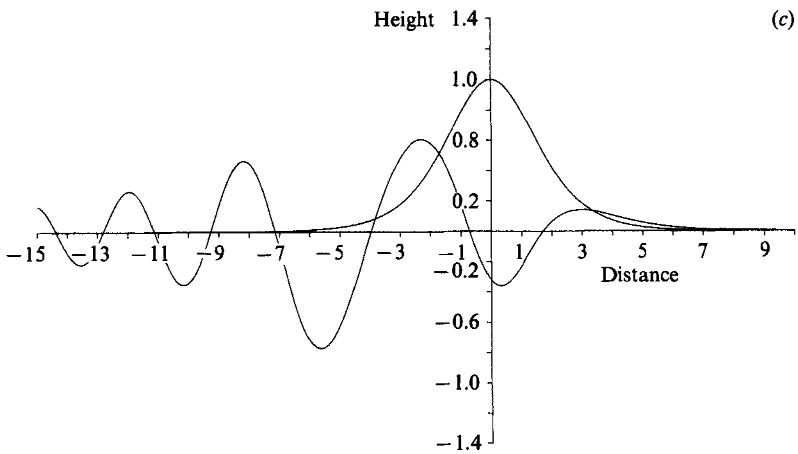


FIGURE 2(*a*–*c*). For caption see p. 517.

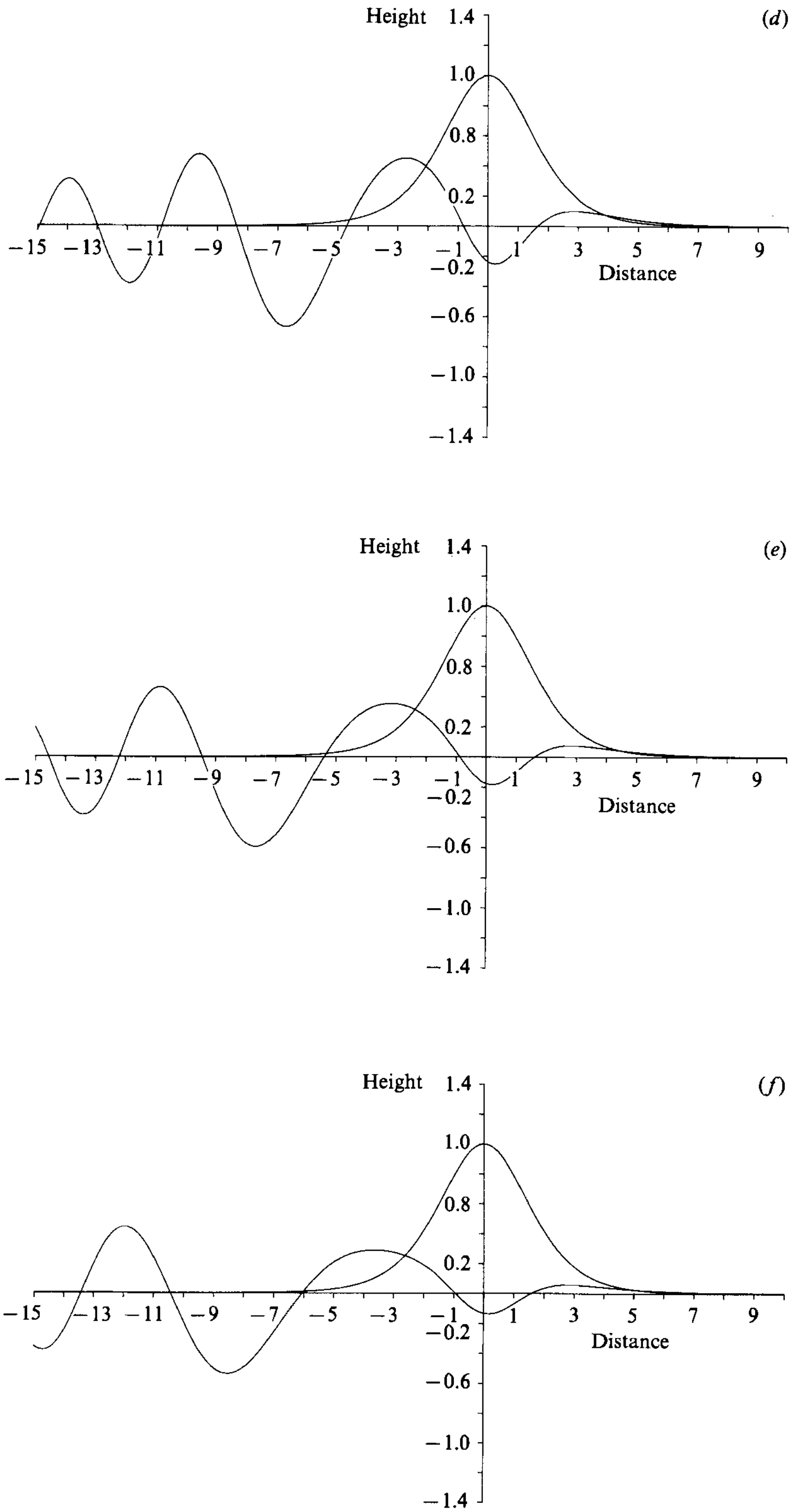


FIGURE 2(*d–f*). For caption see facing page.

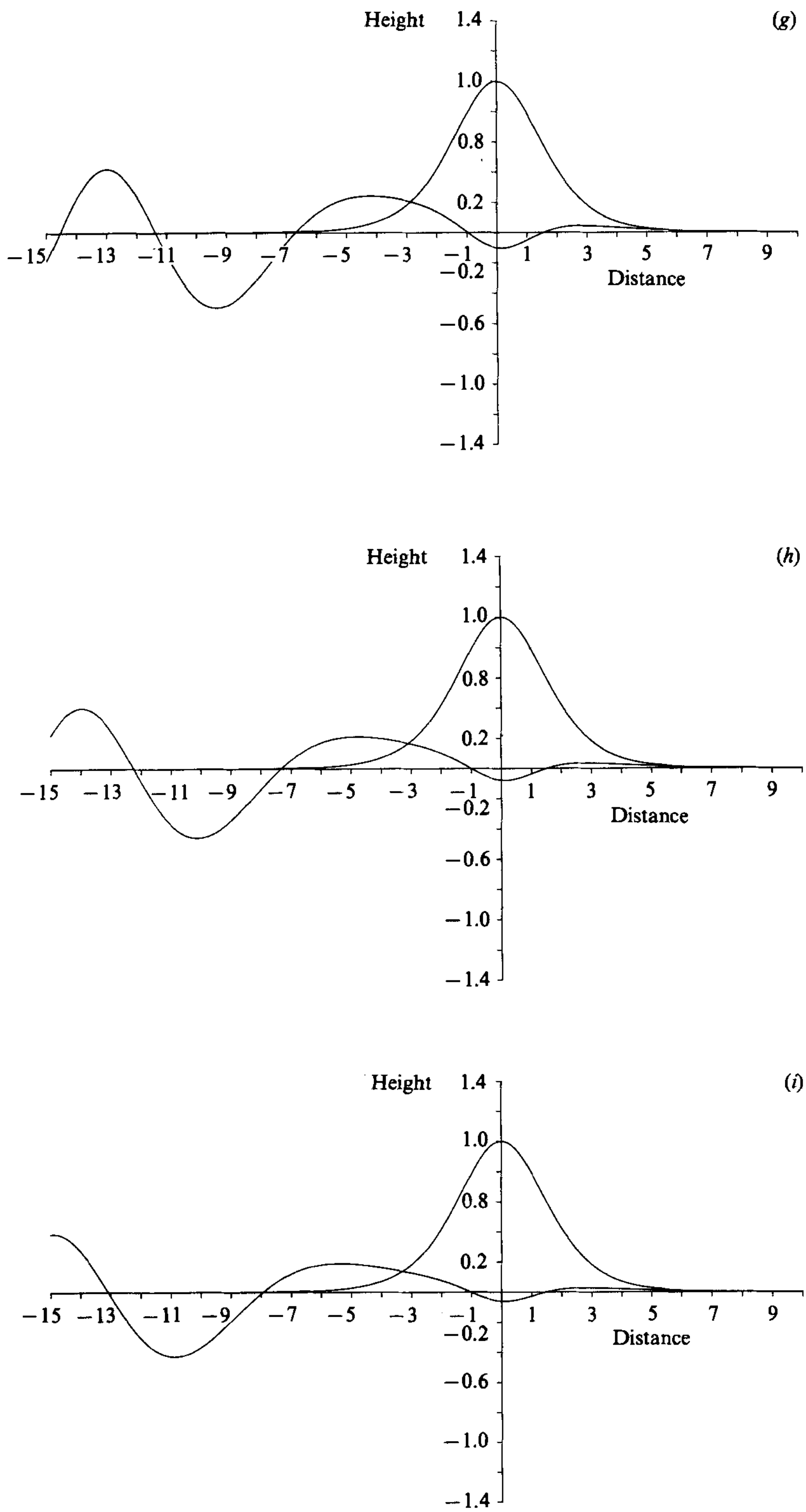


FIGURE 2(*a*–*i*). The formation of the dispersive tail $\tilde{u}_1(\xi_1, \tau_2)$ given by (6.3) for $\tau_2 = 0$–8. Also shown is the solitary wave profile $\mathrm{sech}^2(\frac{1}{2}\xi_1)$.

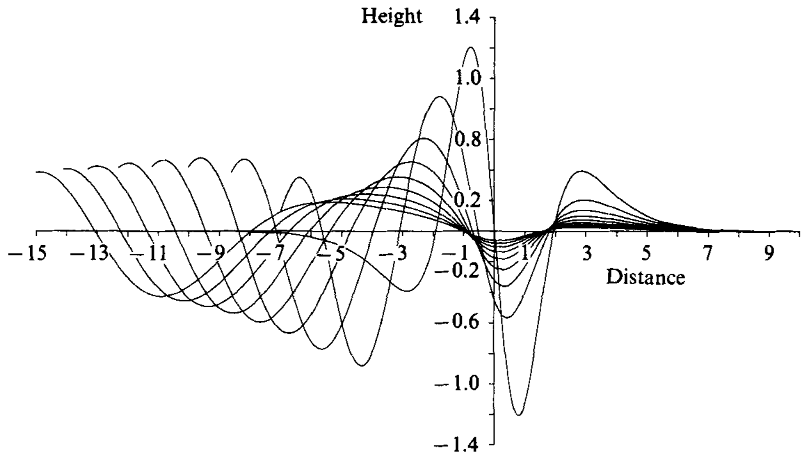


FIGURE 3. The superposition of figure 2(*a–i*).

Using the expansion of (6.5) we have

$$\delta a(\tau_2) \sim \tfrac{4}{9}\epsilon^3(2\pi)^{\frac{1}{2}}\tau_2^{-\frac{1}{2}}3^{-\frac{1}{4}}\operatorname{cosec}(\pi/\sqrt{3})\,\mathrm{e}^{-\tau_2/(3\sqrt{3})} \quad \text{as } \tau_2 \to \infty \tag{6.12}$$

This establishes the result that the effect of the dispersive tail is to produce an apparent third-order change in the amplitude after reflection which decays only on the slow timescale τ_2. The transient loss given by (6.11) is given in figure 1, the factor ϵ^3 being omitted. It should, however, be remembered that (6.11) is only valid as $\xi_2 \to \infty$ with τ_2 fixed.

The formation of the dispersive tail is shown in the sequence of figure 2(*a–i*) for values of $\tau_2 = 0$–8 which are superposed in figure 3. This latter figure clearly shows that the peak of the 'initial' disturbance decreases so that it is the second peak of figure 2(*b*) ($\tau_2 = 1$) which eventually becomes the leading wave of the dispersive tail. The first peak is produced in the exponentially decreasing portion of the dispersive tail and is due to the presence of the two zeros of the multiple of the exponential described by (6.8) and (6.9). Another observation worth noting is that when the solitary wave and dispersive wave are superposed the point of maximum elevation of the combined wave clearly occurs in the range $\xi_1 < 0$ although it increases to zero as $\tau_2 \to \infty$. Thus any measurement of velocity using the combined wave will appear to be greater than the velocity of the solitary wave itself.

Appendix

In this Appendix we prove that to leading order

$$[\kappa] \equiv \int_{-\infty}^{\infty} \tfrac{1}{2}\epsilon \frac{\mathrm{d}\kappa}{\mathrm{d}\tau_2}\,\mathrm{d}\xi_2 = \frac{-5\epsilon^4}{42}. \tag{A 1}$$

From the definition of $\mathrm{d}\kappa/\mathrm{d}\tau_2$ (5.9) we define two integrals

$$I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(18u_0+4)\,\partial_1 u_0\,\tilde{v}_1\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2, \tag{A 2}$$

and

$$I_2 = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (18u_0+4)\,\partial_1 u_0\,\tilde{u}_1\,v_0\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2. \tag{A 3}$$

Using (5.6) we may write I_1 as

$$I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\xi_0=\xi_1}^{\infty} \{u_0(\xi_1)(9u_0(\xi_1)+2)\,\partial_1 u_0(\xi_1)\,u_0(\xi_1)\,\bar{v}_0(k)_2(\xi_2,k)$$
$$\times\, e^{-\frac{1}{4}i\mu\epsilon(\xi_1-\xi_0)+ik\xi_2}\}\,d\xi_0\,dk\,d\xi_1\,d\xi_2. \qquad (A\ 4)$$

Now under the transformation

$$(\xi_0, k, \xi_1, \xi_2) \to -(\xi_0, k, \xi_1, \xi_2), \qquad (A\ 5)$$

the integrand remains unaltered and the only limits that are changed are in the ξ_0 integration, which becomes an integral from $-\infty$ to ξ_1. Adding this form of I_1 to (A 4) gives the result

$$I_1 = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\quad\}\,d\xi_0\,dk\,d\xi_1\,d\xi_2, \qquad (A\ 6)$$

where the integrand is the same as that of (A 4). We now use the fact that the integrand is exponentially small as $\xi_1 \to \infty$, $\xi_0 \to \infty$ or $k \to \infty$. Hence I_1 is unchanged to order ϵ if we replace the factor $e^{-\frac{1}{4}i\mu(\xi_1-\xi_0)}$ by 1. Then I_1 may be expressed as the product of three integrals, J_1, J_2 and J_3, where

$$J_1 = \frac{1}{2}\int_{-\infty}^{\infty} u_0(9u_0+2)\,\partial_1 u_0\,d\xi_1, \qquad (A\ 7)$$

$$J_2 = \int_{-\infty}^{\infty} u_0\,d\xi_0, \qquad (A\ 8)$$

and

$$J_3 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{v}_0(k)\,F_2(\xi_2,k)\,e^{ik\xi_2}\,dk\,d\xi_2. \qquad (A\ 9)$$

Since $J_1 \equiv 0$ we obtain the result

$$I_3 = 0 \quad \text{to } O(\epsilon). \qquad (A\ 10)$$

If this process is repeated on the integral I_2 we obtain

$$I_2 = \tfrac{1}{2}J_4^2 J_5 + O(\epsilon), \qquad (A\ 11)$$

where

$$J_4 = \int_{-\infty}^{\infty} v_0(\xi_2)\,d\xi_2 \equiv -2, \qquad (A\ 12)$$

and

$$J_5 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (9u_0(\xi_1)+2)\,\partial_1 u_0(\xi_1)\,\bar{u}_0(k)\,F_1(\xi_1,k)\,e^{ik\xi_1}\,d\xi_1\,dk. \qquad (A\ 13)$$

The Fourier integral theorem is now used to perform the k-integration as

$$\int_{-\infty}^{\infty}\bar{u}_0(k)\,F_1(\xi_1,k)\,e^{ik\xi_1}\,dk \equiv \int_{-\infty}^{\infty}\bar{u}_0(k)\left\{ik(k^2-1)-4iku_0-2\,\partial_1 u_0+2k^2\frac{\partial_1 u_0}{u_0}\right\}e^{ik\xi_1}\,dk$$
$$= -\partial_1^3 u_0-\partial_1 u_0-4u_0\,\partial_1 u_0-2u_0\,\partial_1 u_0-2\,\partial_1^2 u_0\,\partial_1 u_0/u_0. \qquad (A\ 14)$$

The definition of u_0 shows that

$$\partial_1^3 u_0 = 6u_0\,\partial_1 u_0+\partial_1 u_0, \quad \partial_1^2 u_0 = 3u_0^2+u_0. \qquad (A\ 15)$$

Hence

$$\int_{-\infty}^{\infty}\bar{u}_0(k)\,F_1(\xi_1,k)\,e^{ik\xi_1}\,dk = -18u_0\,\partial_1 u_0-4\,\partial_1 u_0, \qquad (A\ 16)$$

so that

$$J_5 = -\frac{1}{2}\int_{-\infty}^{\infty} (18u_0+4)^2 (\partial_1 u_0)^2 \,\mathrm{d}\xi$$
$$= -\frac{1}{8}\int_{-\infty}^{\infty} (4-9\,\mathrm{sech}^2(\tfrac{1}{2}\xi_1))^2\,\mathrm{sech}^2(\tfrac{1}{2}\xi_1)\tanh^2(\tfrac{1}{2}\xi_1)\,\mathrm{d}\xi_1$$
$$= -\frac{1}{4}\int_{-\infty}^{\infty} (4-9\,\mathrm{sech}^2 x)^2\,\mathrm{sech}^2 x(1-\mathrm{sech}^2 x)\,\mathrm{d}x. \tag{A 17}$$

The integrals S_n, defined by

$$S_n = \int_{-\infty}^{\infty} \mathrm{sech}^{2n} x\,\mathrm{d}x, \quad n \geqslant 1, \tag{A 18}$$

are easily determined by the recurrence relation

$$S_n = \frac{2(n-1)}{2n-1}S_{n-1}, \quad n \geqslant 2 \quad \text{with } S_1 = 2, \tag{A 19}$$

so that

$$J_5 = -20/21. \tag{A 20}$$

Using the above results in the definition of $\mathrm{d}\kappa/\mathrm{d}\tau_2$ (5.9) we obtain

$$[\kappa] = \tfrac{1}{2}\epsilon\int_{-\infty}^{\infty}\frac{\mathrm{d}\kappa}{\mathrm{d}\tau_2}\,\mathrm{d}\xi_2$$
$$= \frac{\epsilon^4}{16}(I_1+I_2)$$
$$= \frac{\epsilon^4}{32}J_4^2 J_5 + O(\epsilon^5)$$
$$= -\frac{5\epsilon^2}{42} + O(\epsilon^5). \tag{A 21}$$

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